MOTION OF A GAS WITH A SPECIFIED THREE-DIMENSIONAL PRESSURE DISTRIBUTION

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V. E. NEUVAZHAEV
(Cheliabinsk)
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The one-dimensional motion of an ideal gas is considered under the action of a distributed pressure, specified according to a power law as a function of mass $p = p_0 m^{-l}$, where $l > 0$. Problems concerning dispersion and symmetrical motion are studied. These problems are formally self-similar for any exponent l. However, when $l > 1$, the system of ordinary differential equations obtained has no solution which satisfies the given boundary conditions. This is due to the fact that in this case, in the vicinity of the point $m = 0$, infinite energy is concentrated. If it be assumed finite by changing the normal pressure distribution in the vicinity of $m = 0$, such a problem is not self-similar but within the limit $t \to \infty$ it tends asymptotically to a self-similar solution with an exponent, which corresponds in the case of dispersion, to the problem of a concentrated shock [1, 2]. For the problem of symmetrical motion, the exponent is always equal to unity and is independent of the initial distribution with $l > 1$.

For the case $l \leqslant 1$, the problem has a self-similar solution for any exponent $l \leqslant 1$. It should be noted that in the case of $l = 1$, the same self-similar exponent is obtained as in the problem concerning a two-dimensional explosion [3]. The main attention is paid below to the case when $l \leqslant 1$. The results of the investigation are presented in the form of graphs. In certain cases, an exact analytical solution has been successfully found for the equations of self-similar motion. The assumption concerning the outcome in a self-similar solution for $l > 1$ is verified by numerical integration of the initial equations in partial derivatives.

1. Suppose we have an ideal gas with equations of state

$$p = R\rho T, \qquad \varepsilon = p/(\gamma - 1)\rho \tag{1.1}$$

where ρ is the density, T the temperature, p the pressure and ε the internal energy; R and γ are the gas constants. For $t = 0$ the gas is at rest,

$$u(0, m) = 0 \tag{1.2}$$

the initial density is constant, $\rho = \rho_0$, and the pressure is given by the power law

$$p(0, m) = p_0 m^{-l} \qquad (l, p_0 = \text{const} > 0) \tag{1.3}$$

where m is the mass coordinate. We shall examine the problems of:

dispersion, when for $t = 0$ the gas is contiguous with a vacuum

$$m = 0, \qquad p(0, t) = 0 \tag{1.4}$$

and symmetrical motion with center at the point $m = 0$

$$m = 0, \qquad u(0, t) = 0 \tag{1.5}$$

The motion of the gas is described in Lagrangian coordinates by the system

$$\frac{\partial}{\partial t}\frac{1}{\rho} - \frac{\partial u}{\partial m} = 0, \qquad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial m} = 0, \qquad \frac{\partial \varepsilon}{\partial t} + p\frac{\partial}{\partial t}\frac{1}{\rho} = 0 \tag{1.6}$$

The conditions in the shock wave have the form

$$\rho_1 (D - u_1) = \rho_0 (D - u_0), \qquad p_1 + \rho_1 (D - u_1)^2 = p_0 + \rho_0 (D - u_0)^2$$
$$\varepsilon_1 - \varepsilon_0 = \frac{1}{2} (p_1 + p_0) \left(\frac{1}{\rho_0} - \frac{1}{\rho_1}\right) \tag{1.7}$$

where D is the shock wave velocity, the subscript zero refers to the state ahead of the wave front and the subscript unity refers to the state behind the wave front.

We introduce dimensionless functions by the formulas (1.8)

$$p = (\gamma - 1) At^{-2l/(l+2)} \rho_0 \pi(\alpha), \qquad \rho = \rho_0 \delta(\alpha), \qquad u = \sqrt{A\gamma(\gamma-1)}\, t^{-l/(l+2)} \zeta(\alpha)$$

$$\alpha = \frac{2mt^{-2/(l+2)}}{(l+2)\sqrt{\gamma(\gamma-1)A}\rho_0}, \qquad p_0 = \left(\frac{l+2}{2}\right)^l \gamma^{1/2l} [A\rho_0(\gamma - 1)]^{1/2(l+2)} \tag{1.9}$$

Substitution of (1.8) and (1.9) into (1.6) and (1.7) gives a system of ordinary differential equations
$$\delta^2\zeta' - \alpha\delta' = 0, \qquad \alpha\zeta' - \frac{1}{\gamma}\pi' = -\frac{1}{2} l\zeta$$
$$-\gamma\pi\delta' + \delta\pi' = -l\pi\delta/\alpha \tag{1.10}$$

with the following conditions in the shock wave

$$\delta_1 (\alpha - \xi_1) = \delta_0 (\alpha - \zeta_0), \quad \gamma^{-1}\pi_1 + \delta_1 (\alpha - \zeta_1)^2 = \gamma^{-1}\pi_0 + \delta_0 (\alpha - \zeta_0)^2$$
$$\frac{\pi_1}{\delta_1} - \frac{\pi_0}{\delta_0} = \frac{\gamma - 1}{2} (\pi_1 + \pi_0) \left(\frac{1}{\delta_0} - \frac{1}{\delta_1}\right) \tag{1.11}$$

The latter equation of system (1.10) can be integrated [3] and we obtain

$$\delta = k\alpha^{l/\gamma}\pi^{1/\gamma} \tag{1.12}$$

where k is the constant of integration. Using Eq. (1.12), we transform system (1.10) into the form

$$\pi' = -\frac{l\gamma}{\Gamma}\left(\frac{\pi}{\gamma} + \frac{k}{2}\alpha^{(l-\gamma)/\gamma}\pi^{(\gamma+1)/\gamma}\zeta\right), \qquad \zeta' = -\frac{l}{\Gamma}\left(\frac{\pi}{\gamma\alpha} + \frac{\zeta}{2}\right) \tag{1.13}$$
$$\Gamma = \alpha - k\alpha^{(l-\gamma)/\gamma}\pi^{(\gamma+1)/\gamma}$$

Following [4] we introduce the new variables Π and Z

$$\zeta = \alpha^{\frac{\gamma-l-1}{\gamma+1}} \Pi^{\frac{\gamma}{\gamma+1}} Z, \qquad \pi = \alpha^{\frac{2\gamma-l}{\gamma+1}} \Pi^{\frac{\gamma}{\gamma+1}} \tag{1.14}$$

System (1.13) then reduces to the following:

$$\alpha\frac{d\Pi}{d\alpha} = \Pi\frac{-(l+2) - {}^1\!/_2 l(\gamma+1)k\Pi Z + (2 - l/\gamma)k\Pi}{1 - k\Pi} \tag{1.15}$$
$$\alpha\frac{dZ}{d\alpha} = \frac{-l/\gamma + {}^1\!/_2(l+2)Z - k\Pi Z + {}^1\!/_2 l\gamma k\Pi Z^2}{1 - k\Pi}$$

Dividing one equation by the other, we obtain

$$\Pi\frac{dZ}{d\Pi} = \frac{-l/\gamma + {}^1\!/_2(l+2)Z - k\Pi Z + {}^1\!/_2 l\gamma k\Pi Z^2}{-(l+2) - {}^1\!/_2 l(\gamma+1)k\Pi Z + (2 - l/\gamma)k\Pi} \tag{1.16}$$

2. In the case of dispersion of the gas into a vacuum, the structure of the solution will be as follows: for $t > 0$ the whole mass of gas is attributed to the motion, except for a point at infinity where, naturally, we set

$$u(\infty, t) = 0 \tag{2.1}$$

The gas-vacuum boundary moves into the vacuum with infinite velocity. A shock wave arises in the gas which propagates through a perturbed background.

Let us set the boundary conditions for Eq. (1.16). First of all, we consider the conditions at infinity (2.1). We note that

$$p = p_0 m^{-l} \alpha^l \pi(\alpha), \qquad u = (\gamma p_0)^{1/2} m^{-1/2 l} \alpha^{1/2 l} \zeta(\alpha) \tag{2.2}$$

follows from Eqs. (1.8) and (1.9).

Hence, with a fixed value of m and $t \to 0$, the initial conditions (1.2) and (1.3) should be valid; therefore

$$\alpha \to \infty, \qquad \alpha^{1/2 l} \zeta(\alpha) \to 0, \qquad \alpha^l \pi(\alpha) \to 1 \tag{2.3}$$

From (1.14) we obtain

$$\Pi \alpha^{l+2} \to 1 \qquad \text{for } \alpha \to \infty \tag{2.4}$$

We shall show that the point

$$\Pi = 0, \qquad Z = 2l/(l+2)\gamma \tag{2.5}$$

must correspond to conditions at infinity.

On the basis of (2.4), the boundary condition for (1.16) lies on the coordinate axis $\Pi = 0$, but as the axis $\Pi = 0$ is the solution of Eq. (1.16), it cannot be the required solution and then the starting point will be singular, i. e. the solution intersects the straight line $\Pi = 0$. Equation (2.5) and the point

$$\Pi = 0, \qquad z = \infty \tag{2.6}$$

will be singular points belonging to the axis $\Pi = 0$.

The latter cannot be the starting point. It is clear from physical considerations, that in the vicinity of the point (2.6), the functions Π and Z are positive (cf. Eq. (1.8) and (1.14)). It is easy to show that Eq. (1.16) has an entire set of solutions for $\Pi > 0$ and $Z > 0$, emerging from the point (2.6)

$$\Pi = CZ^{-2} + \ldots \tag{2.7}$$

where C is some constant. However, in this case we have

$$\zeta = C_1 \alpha^{-1/2 l} + \ldots$$

in the vicinity of the point being considered.

This contradicts the condition $\zeta \alpha^{-1/2 l} \to 0$ for $\alpha \to \infty$ from Eq. (2.3). Therefore, the unique point (2.5) corresponds to the conditions at infinity in the variables Π and Z. It has the nature of a saddle and the required solution has the following expansion:

$$\Pi = -\frac{3\gamma (l+2)^3}{4l(l^2 - l - 2)} \left(z - \frac{2l}{(l+2)\gamma} \right) + \ldots \tag{2.8}$$

3. We shall consider condition (1.4) at the gas-vacuum boundary and we shall show that in the variables Π and Z, a singular point with the coordinates

$$\Pi_0 = 0, \qquad Z_0 = -\infty \tag{3.1}$$

corresponds to this condition.

We shall find also the expansion of the solution in the vicinity of (3.1).

Let us suppose that Π_0 and Z_0 are finite values for $\alpha = 0$. The equations for determining Π_0 and Z_0 follow then from (1.14) and (1.13)

$$-\frac{2\gamma - l}{(\gamma+1) l \gamma} (1 - k\Pi_0) = \frac{1}{\gamma} + \frac{k}{2} \Pi_0 Z_0$$

$$-\frac{\gamma - l - 1}{l(\gamma+1)} Z_0 (1 - k\Pi_0) = \frac{1}{\gamma} + \frac{1}{2} Z_0$$

Solving the latter, we have

$$A\left(Z_0 = -\frac{2}{\gamma},\ \Pi_0 = \frac{1}{k}\right), \qquad B\left(Z_0 = \frac{4(l-2\gamma)}{\gamma(l+2)(1-\gamma)},\ \Pi_0\right)$$

For the point B with $l - 2\gamma < 0$ we obtain $Z_0 > 0$, and therefore this point must be discarded, as it follows from (1.14) that $Z_0 < 0$ for $\alpha = 0$.

The point A, just as point B, is singular for Eq. (1.16). Its character of singularity is a saddle. The separatrixes of the saddle have the expansions

$$\Pi = \frac{1}{k} + \lambda_1\left(Z + \frac{2}{\gamma}\right) + \ldots, \qquad \Pi = \frac{1}{k} + \lambda_2\left(Z + \frac{2}{\gamma}\right) + \ldots \quad (3.2)$$

Substituting (3.2) into (1.15), we obtain

$$\Pi = (l+2)\ln\alpha + C \quad (3.3)$$

This contradicts the assumption that Π_0 is finite. Consequently, the point A also should be discarded. There remains the unique point (3.1), which, in fact, corresponds to the gas-vacuum boundary. The expansion of the solution in its vicinity has the form

$$\Pi = -\frac{2+l}{lk}\frac{1}{Z} + \ldots, \qquad Z = C\alpha^{1/2(l+2)(1-\gamma)} + \ldots \quad (3.4)$$

$$\zeta \sim \alpha^{-1/2 l}, \qquad \pi \sim \alpha^{1/2(l+2)\gamma - l} \quad (3.5)$$

A further set of integral curves (2.7) results from (3.1), for which $\pi \sim \alpha^{-l}$; this however, for $l > 0$, contradicts the boundary condition (1.4). Therefore, the unknown solution is represented by the expansion (3.4). It can be seen from (3.5) that the gas flows out with infinite velocity. The behavior of the integral curves in the vicinity of (3.1) is shown in Fig. 1.

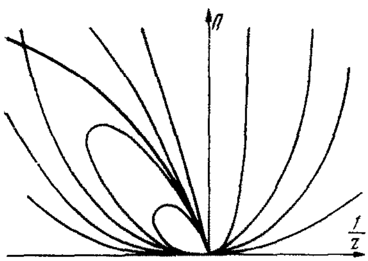

Fig. 1

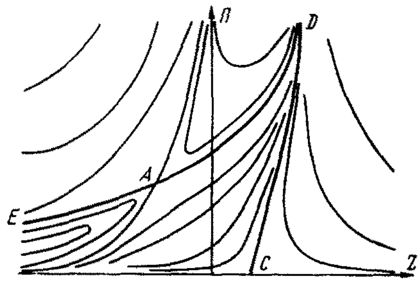

Fig. 2

4. We shall describe the method for the construction of the solution. From the foregoing, the solution must lie in the upper half-plane of $\Pi \geqslant 0$ (Fig. 2) and it must pass through the singular points C and E. It is obvious that in moving along the curve CD from point C to point D, the parameter α must decrease. However, this will occur so long as $\Pi < k^{-1}$. Consequently, it is not possible to construct a continuous solution and it is necessary to use the conditions in the shock wave (1.11) in order to arrive at point E, starting from point C. From the conditions in the shock wave we have $Z_1 > 0$ and therefore the solution further must pass through the singular point A.

The constant k occurring in Eq. (1.16) will be different for different sides of the discontinuity. It follows from condition (2.3) that ahead of the shock wave front $k_0 = 1$. Further construction of the solution consists in choosing the location of the shock wave front, defined by the value of α. The latter must be selected so that the point (Π_1, Z_1) obtained from (1.14) and from the conditions in the shock wave (1.11) and (1.12) should lie on the separatrix DA. The five unknowns α, ζ_1, δ_1, π_1 and k_1, generally speaking, are defined from the three conditions (1.11), relation (1.12) and the condition for passage of the integral curve through the singular point A.

Numerical integration of Eq. (1.16) is carried out over the interval $0 < \Pi < 1$, and at the point (2.5) the expansion (2.8) is used. When the function $Z = Z(\Pi)$ is defined, the solution of the first equation of system (1.15) is found in the interval $\alpha_* < \alpha < \infty$, taking account that the point ($\alpha = \infty$, $\Pi = 0$) is singular and the unknown integral curve has the asymptotics (2.4). Then, by formulas (1.11), (1.12) and (1.14) we compute successively

$$\zeta_0 = \alpha^{(\gamma-l-1)/(\gamma+1)}\Pi_0^{\gamma/(\gamma+1)}Z_0, \qquad \delta_1 = \delta_0\frac{\zeta_0-\alpha}{\zeta_1-\alpha}, \qquad \pi_0 = \alpha^{(2\gamma-l)/(\gamma+1)}\Pi_0^{\gamma/(\gamma+1)}$$

$$\delta_0 = \alpha^{l/\gamma}\pi_0^{1/\gamma}, \qquad \zeta_1 = \alpha + \frac{\gamma-1}{\gamma+1}(\zeta_0-\alpha) + \frac{2\pi_0}{(\gamma+1)\delta_0(\zeta_0-\alpha)}, \qquad k_1 = \delta_1\alpha^{-l/\gamma}\pi_1^{-1/\gamma}$$

$$\pi_1 = \pi_0\frac{1-\gamma}{1+\gamma} + \frac{2\gamma}{1+\gamma}\delta_0(\zeta_0-\alpha)^2, \qquad \Pi_1 = \alpha^{(l-2\gamma)/\gamma}\pi_1^{(\gamma+1)/\gamma} \tag{4.1}$$

$$Z_1 = \alpha^{(l+1-\gamma)/(\gamma+1)}\Pi_1^{-\gamma/(\gamma+1)}\zeta_1$$

The curve $\Pi_1(Z_1)$ obtained should intersect the separatrix EAD, also defined numerically, for a certain value of the parameter α_1.

5. Let us consider the case $l > 1$. We note that intersection is not always possible. This implies that the formulation of the problem does not have a self-similar solution. We shall prove this for $l = 2$ and $\gamma = 1.5$.

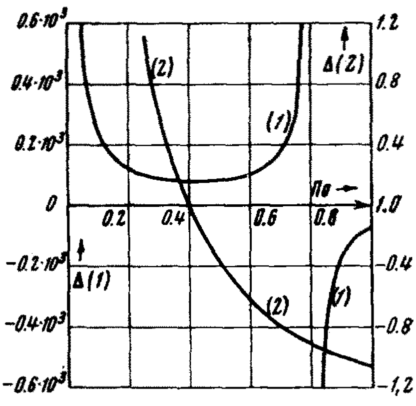

Fig. 3

In the particular case when $l = 2(\gamma - 1)/(2 - \gamma)$, the separatrix DA can be defined analytically

$$\Pi = \frac{1}{k}\cdot 1/(2-\gamma)/\gamma - (\gamma-1)Z \tag{5.1}$$

The integral curve CD for $l = 2$ will be the straight line $Z = 1/\gamma$. For $l = 2$ and $\gamma = 1.5$ we have

$$\alpha = (1 + {}^1/_4\,\Pi_0)^{5/4}\Pi_0^{-1/4}$$

By formula (4.1) we obtain a certain curve

$$\Delta = \pi_1^{5/3}\alpha^{-2/3} - 6/k\,(3 - 2\alpha\zeta_1\pi^{-1/4})$$

which is plotted in Fig. 3. It can be seen from the graph that it does not intersect the axis $\Delta = 0$ and, therefore, for $\gamma = 1.5$ and $l = 2$, there is no self-similar solution for the problem being considered.

In order to explain the physical essence of the solution obtained in this case, we determine the total energy of the system at $t = 0$

$$E_0 = \frac{p_0}{(\gamma-1)\rho_0}\frac{m^{1-l}}{1-l}\Bigg|_0^\infty \tag{5.2}$$

We find that it is infinite: the integral diverges at the point $m = 0$ for $l > 1$ and at $m = \infty$ for $l < 1$. We also infer that if $l > 1$, then a finite gas mass has infinite energy for some fixed m_0, while a gas of infinite mass ($m > m_0$) has finite energy. In this case an infinite amount of energy is concentrated at the boundary between the gas and a vacuum. If we assume that this energy is finite, e.g. by setting

$$p = p_0 m_0^{-l} \quad (m \leqslant m_0), \qquad p = p_0 m^{-l} \quad (m > m_0) \tag{5.3}$$

then the problem is no longer self-similar. The self-similar solution onto which the problem will emerge is not readily apparent in advance. It might be some solution of system of ordinary equations (1.15). But, as we showed in the case $\gamma = 1.5$, $l = 2$, such a solution does not exist. Numerical calculations by a difference procedure [5] carried out for initial partial differential equations (1.6) indicate a fairly rapid emergence onto a self-similar solution with an exponent equal to that of the concentrated shock problem [1].

The dependence of the coordinate of the shock wave front on time is plotted in Fig. 4 for $\gamma = 1.4$ and $l = 2$. The solid line denotes the wave front in the self-similar problem of concentrated shock through a cold gas ($l = {}^4/_3$, see [6]), and the points refer to the results of numerical integration of the non-self-similar problem with (5.3) for $l = 2$ and $\gamma = 1.4$.

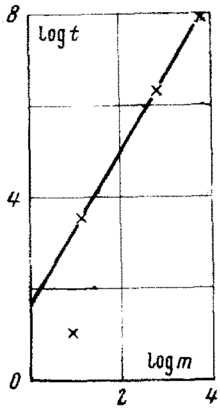

Fig. 4

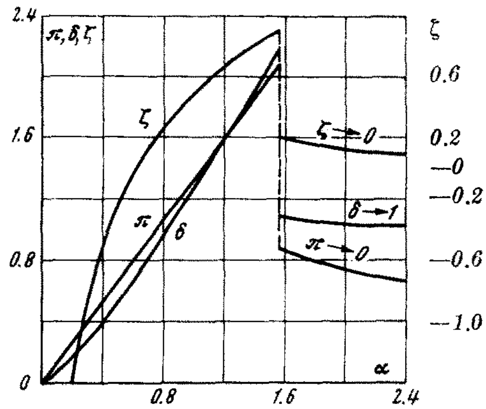

Fig. 5

6. The case $l \leqslant 1$. The problem is self-similar for this distribution of the initial pressure. Curve 2 in Fig. 3 intersects the axis $\Delta = 0$. The method for constructing the solution is shown in Sect. 3.

With $l = 2(\gamma - 1) / (2 - \gamma)$, the solution behind the shock wave front can be written in the finite form

$$\pi = \frac{\pi_1}{\alpha_1}\alpha, \qquad \delta = k_1 (\pi_1/\alpha_1)^{1/\gamma} \alpha^{1/(2-\gamma)} \tag{6.1}$$

$$\zeta = (\pi_1/\alpha_1)^{-1/\gamma} \frac{1}{(1-\gamma)k_1} \alpha^{(\gamma-1)/(\gamma-2)} + \frac{2-\gamma}{(\gamma-1)\gamma}\frac{\pi_1}{\alpha_1}$$

The solution ahead of the front is obtained as a result of numerical integration of Eqs. (1.15) and (1.16). It is shown for $\gamma = 1.2$ and $l = 0.5$ in Fig. 5. At the shock wave front, the corresponding parameters of the problem have been obtained as

$$\alpha_1 = 1.56, \quad \delta_0 = 1.09, \quad \delta_1 = 2.19, \quad \pi_0 = 0.892$$
$$\pi_1 = 2.08, \quad \zeta_0 = 0.213, \quad \zeta_1 = 0.886, \quad k_1 = 0.988$$

It can be seen from Fig. 5 that the maximum values of the density, velocity and pressure are achieved at the shock wave front. The compression in the shock wave is found to be significantly below the limiting value, which is equal to 11 for $\gamma = 1.2$.

7. Let us consider the symmetrical motion of a gas. This problem admits generalization in cylindrical and spherical geometry. We shall investigate the point corresponding to the plane of symmetry $m = 0$. The boundary condition (1.5) assumes the form

$$\alpha = 0, \qquad \zeta = 0 \tag{7.1}$$

If we assume that the pressure in the plane of symmetry is nonvanishing, it follows from Eq. (1.14) that

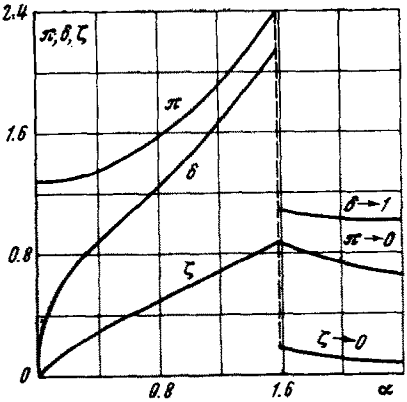

Fig. 6

$$Z = \frac{\zeta\alpha}{\pi}, \qquad Z(\alpha = 0) = 0 \qquad (7.2)$$

and, likewise, when $2\gamma - l > 0$

$$\Pi(\alpha = 0) = \infty \qquad (7.3)$$

Therefore, in the plane ΠZ, the point

$$Z = 0, \qquad \Pi = \infty \qquad (7.4)$$

corresponds to the boundary condition.

This singular point is a node. It can be shown that the required solution is a curve intersecting a set of integral curves at the singular point. The expansion of the solution for $l < \gamma$ has the form

$$\Pi = l/k\,(\gamma - l)\;1/Z + \ldots \qquad (7.5)$$

As in the case of the dispersion problem, the solution here can be constructed for $l \leqslant 1$. If $l = 1$, then behind the shock wave the solution is written in finite form, as the curve

$$\Pi = 2/k\,(\gamma - 1)Z\,(2 - \gamma Z) \qquad (7.6)$$

is the solution (1.16). Knowing (7.6), Eq. (1.15) can be integrated. For $l > 1$, there is no solution for the self-similar problem. If, however, the normal pressure distribution (5.3) is taken as the initial distribution for $l > 1$, this problem emerges as a self-similar solution for $t \to \infty$, corresponding to the two-dimensional explosion problem [3] with the self-similarity exponent $l = 1$. Fig. 6 shows the solution for $\gamma = 1.4$ and $l = 0.5$. A shock wave is propagated through the perturbed background and the maximum values of velocity, density and pressure are attained in it. In the case when $m = 0$, the pressure is finite and the density is equal to zero.

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